MIDTERM: ALGEBRAIC GEOMETRY

Date: 2^{nd} **March 2015**
The Total points is **115** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity.**

(1) (10+5=15 points) Let $\phi : X \to Y$ be a morphism of affine varieties over an algebraically closed field k. If ϕ is surjective show that the induced map of k-algebra $\phi^{\#} : \mathcal{O}_Y \to \mathcal{O}_X$ is injective. Also show that the converse is false.

(2) (5+15=20 points) Let X be an algebraic subset of a projective space $\mathbb{P}^n$. When is X called irreducible? Assuming X to be an algebraic subset of a projective space, show that X is irreducible iff the homogeneous coordinate ring of $X \subset \mathbb{P}^n$ is a domain.

(3) (20 points) Let R be a ring and I an ideal of R. Define minimal primes of R. Compute the irreducible components of the affine algebraic subset of $\mathbb{A}^2_{\mathbb{C}}$ defined by the polynomial $f(x, y) = (x^2 + y^2)(x^2 + y^2 + 1)(x^4 - y^4)$. What are the minimal primes of the ideal (f) in $\mathbb{C}[x, y]$?

(4) (5+15=20 points) Let X be a topological space. What does it mean to say that the sequence of sheaves $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ on X is exact? Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be an exact sequence of sheaves on X. Let U be an open subset of X. Show that the following sequence of abelian groups is exact:

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

(5) (20 points) Let U be an irreducible curve of degree $d \geq 1$ in $\mathbb{A}^2$, i.e. U is the zero set of an irreducible polynomial in two variables of degree d and X be its closure in $\mathbb{P}^2$. Let r be the number of points in X that are not in U. Show that r is between 1 and d.

(6) (20 points) Let A be a ring and I be an ideal of A. Let $X = \text{Spec}(A)$ and $\mathcal{O}_X$ be the structure sheaf. Let $\mathcal{I}$ be the ideal sheaf corresponding to I (i.e. for U open in X, $\mathcal{I}(U) = r_U(I)\mathcal{O}_X(U)$ where $r_U : \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is the restriction map). Show that the stalk of the sheaf $\mathcal{O}/\mathcal{I}$ at $P \in X$ is 0 if P is not in $V(I)$. What is the stalk if $P \in V(I)$?